Kinematics & Dynamics of Linkages

Lecture 12: 2 Positions Analytical Synthesis

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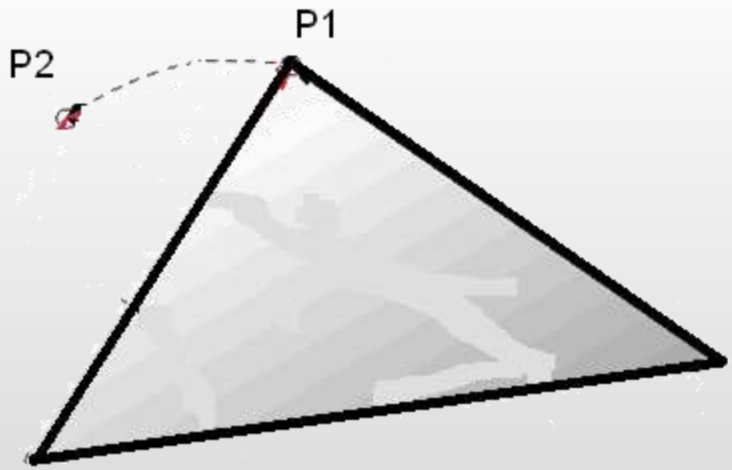
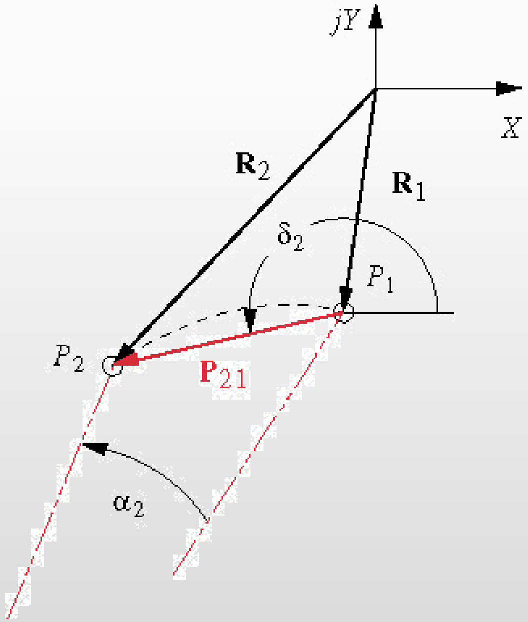
LAU

2-Position Motion Generation

- Design four bar linkage that will move a line on its coupler link such that a point P on that line will be first at P1 and later at P2 and will also rotate the line through an angle α_2

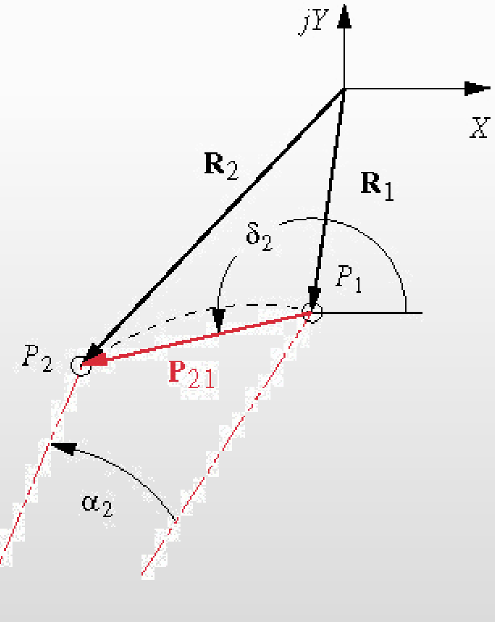




2-Position Procedure

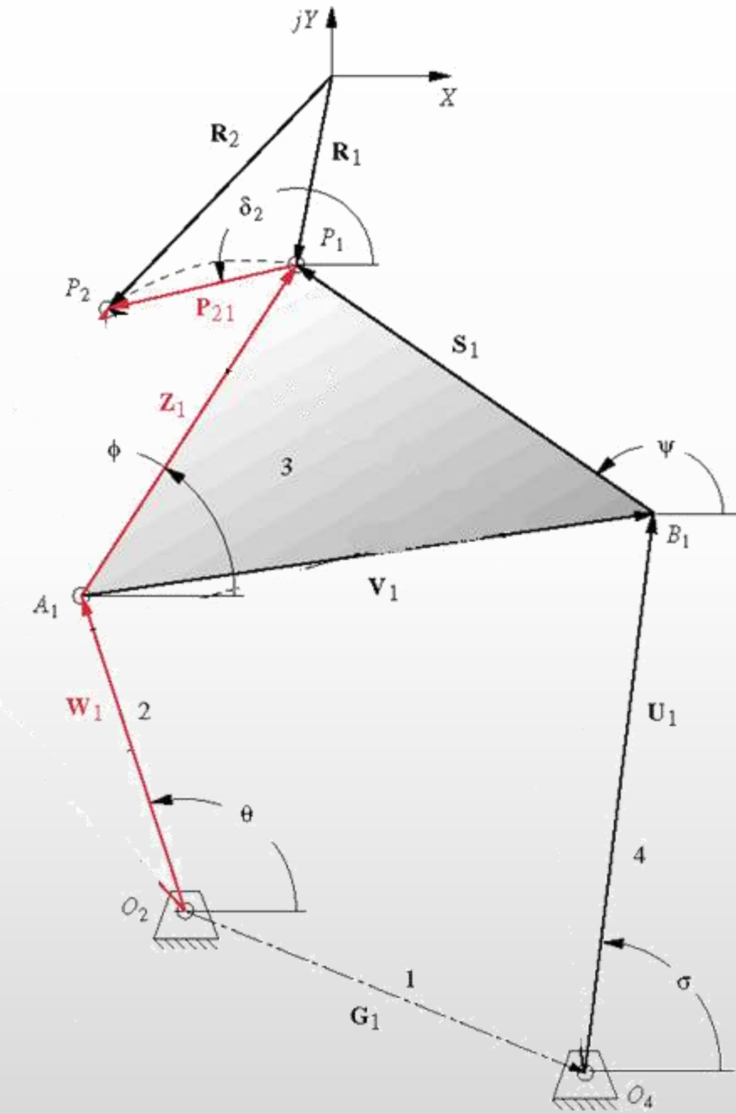
Define the 2 desired precision positions w/r to a chosen GCS

1. Use position vectors R_1 and R_2
2. α_2 is the change in orientation of link z
3. p_{21} defines the distance from point P_1 to P_2
4. $p_{21} = R_2 - R_1$

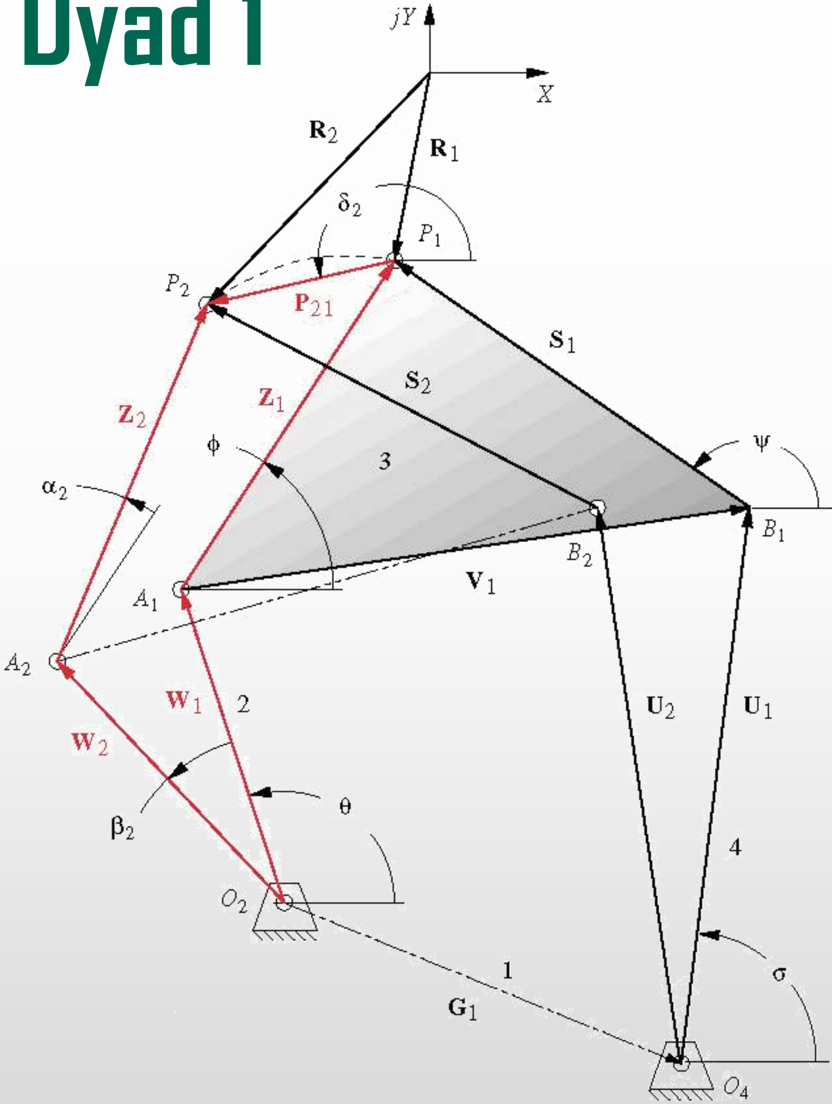


2-Position Procedure

- The dyad $\mathbf{W_1 Z_1}$ defines the left half of the linkage. The dyad $\mathbf{U_1 S_1}$ defines the right half of the linkage.
- Solve for the left side of the linkage (vectors $\mathbf{W_1}$, $\mathbf{Z_1}$) and later use the same procedure to solve for the right side (vectors $\mathbf{U_1}$, $\mathbf{S_1}$).



2-Position Procedure – Dyad 1



1st dyad: WZ

Define the following variables:

w = length link2

θ = angle of link2 in first position

β_2 = change in angle of input link

z = distance from A and P

Φ = orientation angle of L_3 at position 1

α_2 = change in orientation of link z

p_{21} = distance from point P_1 to P_2

δ_2 = orientation angle of line from P_1 to P_2

2-Position Procedure – Dyad 1

Equation of the 1st dyad

$$W_2 + Z_2 - P_{21} - Z_1 - W_1 = 0$$

$$w\,e^{j(\theta+\beta_2)} + z\,e^{j(\Phi+\alpha_2)} - p_{21}\,e^{j\delta_2} - z\,e^{j\Phi} - w\,e^{j\theta} = 0$$

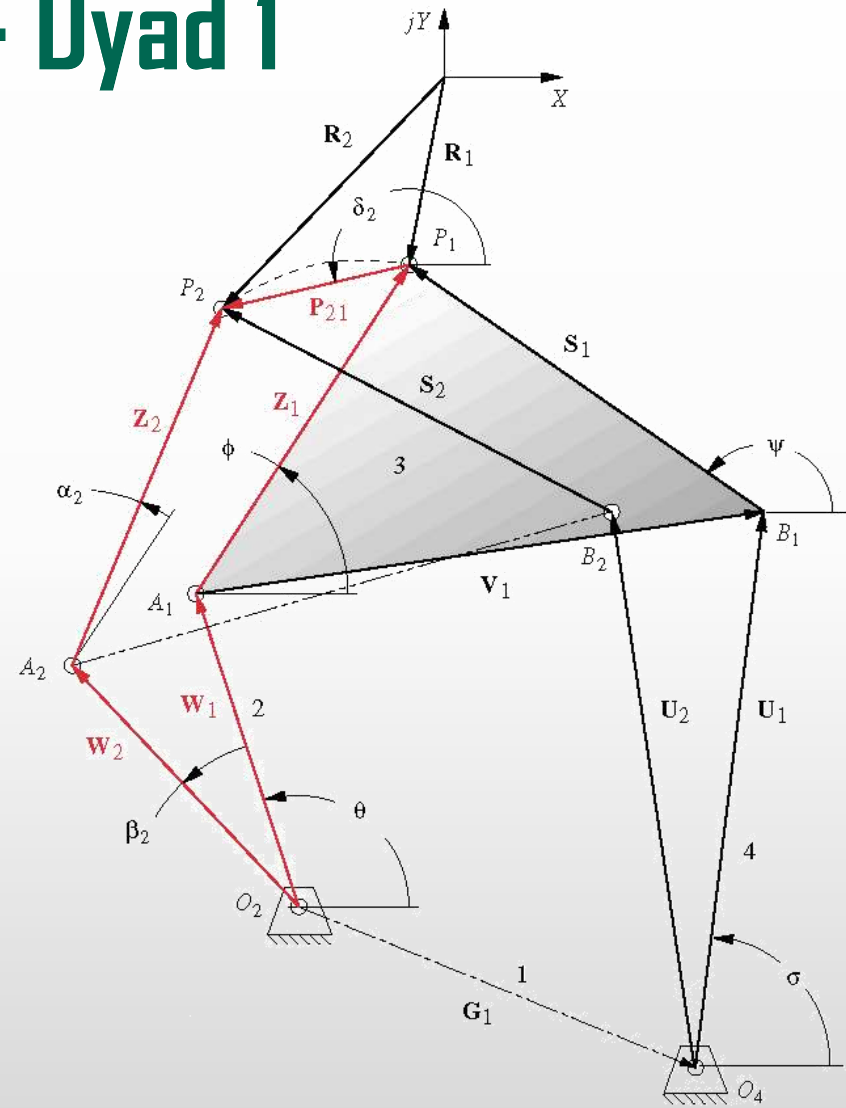
$$w\,e^{j\theta}(e^{j\beta_2} - 1) + z\,e^{j\Phi}(e^{j\alpha_2} - 1) = p_{21}\,e^{j\delta_2}$$

Real part:

$$w\,\mathrm{Cos}\theta\,(\mathrm{Cos}\beta_2 - 1) - w\,\mathrm{Sin}\theta\,\mathrm{Sin}\beta_2$$
$$+ z\,\mathrm{Cos}\Phi(\mathrm{Cos}\alpha_2 - 1) - z\,\mathrm{Sin}\Phi\,\mathrm{Sin}\alpha_2 = p_{21}\,\mathrm{Cos}\delta_2$$

Imaginary part:

$$w\,\mathrm{Sin}\theta\,(\mathrm{Cos}\beta_2 - 1) + w\,\mathrm{Cos}\theta\,\mathrm{Sin}\beta_2$$
$$+ z\,\mathrm{Sin}\Phi(\mathrm{Cos}\alpha_2 - 1) + z\,\mathrm{Cos}\Phi\,\mathrm{Sin}\alpha_2 = p_{21}\,\mathrm{Sin}\delta_2$$



2-Position Procedure – Dyad 1

2 Equations:

- $w\,\text{Cos}\boldsymbol{\theta}\,(\text{Cos}\boldsymbol{\beta}_2 - 1) - w\,\text{Sin}\boldsymbol{\theta}\,\text{Sin}\boldsymbol{\beta}_2 + z\,\text{Cos}\boldsymbol{\Phi}(\text{Cos}\boldsymbol{\alpha}_2 - 1) - z\,\text{Sin}\boldsymbol{\Phi}\,\text{Sin}\boldsymbol{\alpha}_2 = p_{21}\,\text{Cos}\boldsymbol{\delta}_2$
- $w\,\text{Sin}\boldsymbol{\theta}\,(\text{Cos}\boldsymbol{\beta}_2 - 1) + w\,\text{Cos}\boldsymbol{\theta}\,\text{Sin}\boldsymbol{\beta}_2 + z\,\text{Sin}\boldsymbol{\Phi}(\text{Cos}\boldsymbol{\alpha}_2 - 1) + z\,\text{Cos}\boldsymbol{\Phi}\,\text{Sin}\boldsymbol{\alpha}_2 = p_{21}\,\text{Sin}\boldsymbol{\delta}_2$

There are 8 variables in the equations:

w, θ, β_2, z, Φ, α_2, p_{21}, δ_2

3 are given: α_2, p_{21} and δ_2

Choose 3 variables and solve for 2.

2 Cases to be considered:

Choose θ , β_2 and Φ solve for w and z

Choose β_2 ,Φ and z solve for w and θ

2-Position Procedure – Dyad 1: Case 1

Choose θ , β_2 and Φ **and** solve for $\boldsymbol{w}$ and $\boldsymbol{z}$

Let:

$A = \cos\theta\,(\cos\beta_2 - 1) - \sin\theta\sin\beta_2$

$B = \cos\Phi\,(\cos\alpha_2 - 1) - \sin\Phi\sin\alpha_2$

$C = \boldsymbol{p}_{21}\cos\delta_2$

$D = \sin\theta\,(\cos\beta_2 - 1) + \cos\theta\sin\beta_2$

$E = \sin\Phi\,(\cos\alpha_2 - 1) + \cos\Phi\sin\alpha_2$

$F = \boldsymbol{p}_{21}\sin\delta_2$

Substitute in the equations below:

$$\boldsymbol{w}\cos\theta\,(\cos\beta_2 - 1) - \boldsymbol{w}\sin\theta\,\sin\beta_2 + \boldsymbol{z}\cos\Phi(\cos\alpha_2 - 1) - \boldsymbol{z}\sin\Phi\,\sin\alpha_2 = \boldsymbol{p}_{21}\,\cos\delta_2$$

$$\boldsymbol{w}\sin\theta\,(\cos\beta_2 - 1) + \boldsymbol{w}\cos\theta\,\sin\beta_2 + \boldsymbol{z}\sin\Phi(\cos\alpha_2 - 1) + \boldsymbol{z}\cos\Phi\,\sin\alpha_2 = \boldsymbol{p}_{21}\,\sin\delta_2$$

$$A\,\mathbf{w} + B\,\mathbf{z} = C$$

$$D\,\mathbf{w} + E\,\mathbf{z} = F$$

Solve:

$$\boldsymbol{w} = (C\,E - B\,F)/(A\,E - B\,D)$$

$$\boldsymbol{z} = (A\,F - C\,D)/(A\,E - B\,D)$$

2-Position Procedure – Dyad 1: Case 2

Choose β_2, Φ and $\boldsymbol{z}$ solve for $\boldsymbol{w}$ and θ

Let:

$W_{1x} = \boldsymbol{w}\,\text{Cos}\theta$

$W_{1y} = \boldsymbol{w}\,\text{Sin}\theta$

$Z_{1x} = \boldsymbol{z}\,\text{Cos}\Phi$

$Z_{1y} = \boldsymbol{z}\,\text{Sin}\Phi$

And

$A = \text{Cos}\beta_2 - 1$

$B = \text{Sin}\beta_2$

$C = \cos\alpha_2 - 1$

$D = \text{Sin}\alpha_2$

$E = \boldsymbol{p}_{21}\text{Cos}\delta_2$

$F = \boldsymbol{p}_{21}\text{Sin}\delta_2$

Substitute in the equations below:

$$\boldsymbol{w}\,\text{Cos}\theta\,(\text{Cos}\beta_2 - 1) - \boldsymbol{w}\,\text{Sin}\theta\,\text{Sin}\beta_2 + \boldsymbol{z}\,\text{Cos}\Phi(\text{Cos}\alpha_2 - 1) - \boldsymbol{z}\,\text{Sin}\Phi\,\text{Sin}\alpha_2 = \boldsymbol{p}_{21}\,\text{Cos}\delta_2$$

$$\boldsymbol{w}\,\text{Sin}\theta\,(\text{Cos}\beta_2 - 1) + \boldsymbol{w}\,\text{Cos}\theta\,\text{Sin}\beta_2 + \boldsymbol{z}\,\text{Sin}\Phi(\text{Cos}\alpha_2 - 1) + \boldsymbol{z}\,\text{Cos}\Phi\,\text{Sin}\alpha_2 = \boldsymbol{p}_{21}\,\text{Sin}\delta_2$$

$$AW_{1x} - B\,W_{1y} + C\,Z_{1x} - D\,Z_{1y} = E$$

$$AW_{1y} + B\,W_{1x} + C\,Z_{1y} + D\,Z_{1x} = F$$

Solve:

$$W_{1x} = -\,(A(-\,CZ_{1x} + DZ_{1y} + E) + B(-\,CZ_{1y} - DZ_{1x} + F))/2A$$

$$W_{1y} = -\,(A(-\,CZ_{1y} - DZ_{1x} + F) + B(CZ_{1x} - DZ_{1y} - E))/2A$$

2-Position Procedure – Dyad 2

2nd dyad: US

Define the following variables:

u = length of link2

σ = angle of link2 in first position

γ_2 = change in angle of driven link

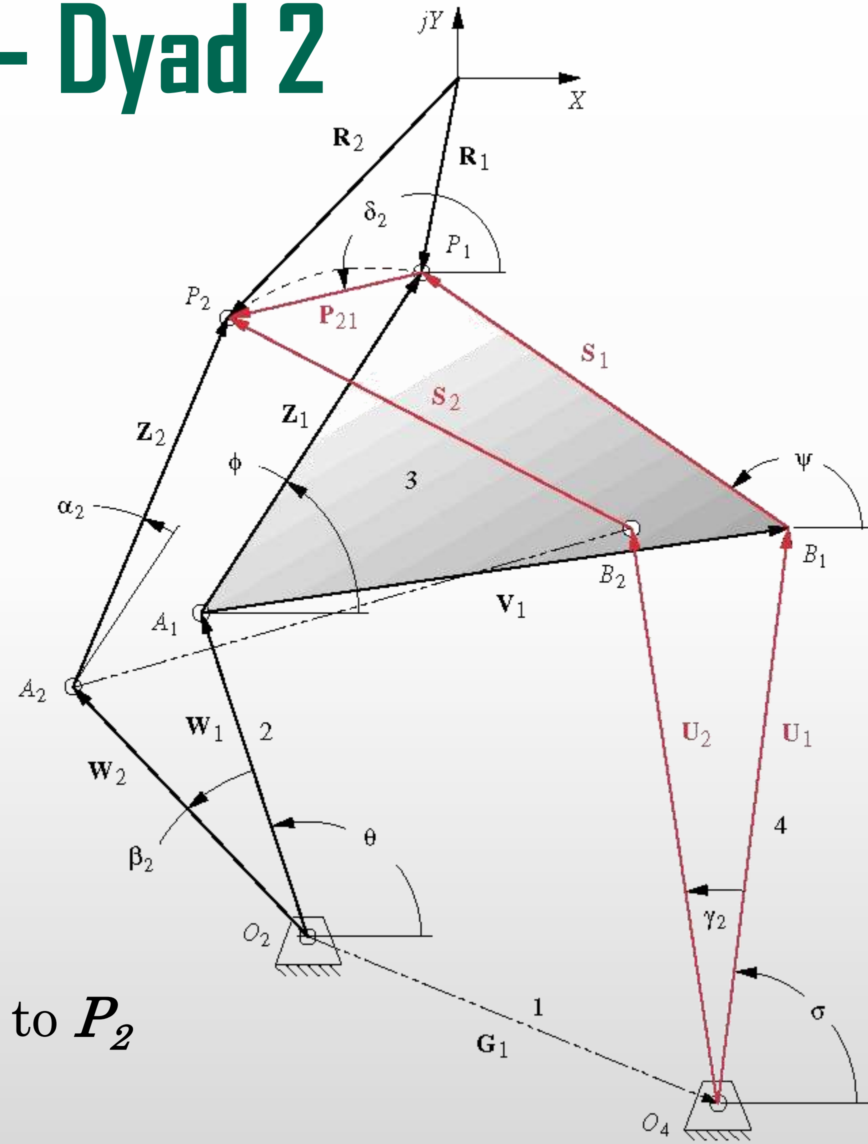
s = distance from point B to point P

Ψ = orientation angle of link s

α_2 = change in orientation of link s

p_{21} = distance from point P_1 to P_2

δ_2 = orientation angle of line from P_1 to P_2



2-Position Procedure – Dyad 2

Equation of the 2nd dyad

$$U_2 + S_2 - P_{21} - S_1 - U_1 = 0$$

$$u\, e^{j\sigma}(e^{j\gamma_2} - 1) + s\, e^{j\Psi}(e^{j\alpha_2} - 1) = p_{21}\, e^{j\delta_2}$$

Real part:

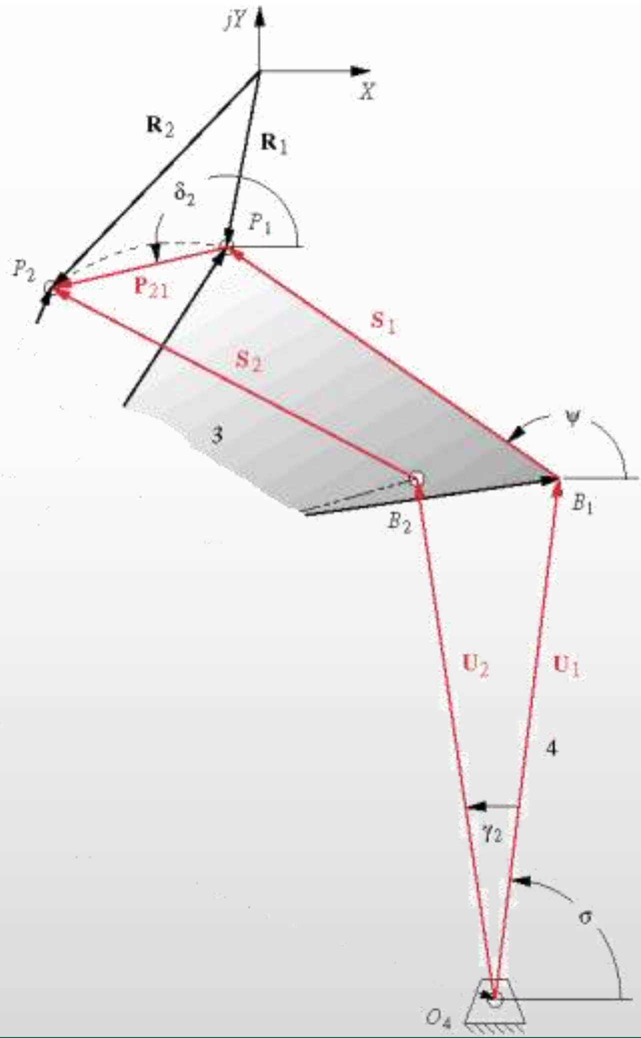
$$u \cos\sigma\,(\cos\gamma_2 - 1) - u \sin\sigma \sin\gamma_2$$

$$+ s \cos\Psi(\cos\alpha_2 - 1) - s \sin\Psi \sin\alpha_2 = p_{21} \cos\delta_2$$

Imaginary part:

$$u \sin\sigma\,(\cos\gamma_2 - 1) + u \cos\sigma \sin\gamma_2$$

$$+ s \sin\Psi(\cos\alpha_2 - 1) + s \cos\Psi \sin\alpha_2 = p_{21} \sin\delta_2$$



2-Position Procedure – Dyad 2

2 Equations:

- $u\,\mathrm{Cos}\,\sigma(\mathrm{Cos}\,\gamma_2 - 1) - u\,\mathrm{Sin}\,\sigma\,\mathrm{Sin}\,\gamma_2 + s\,\mathrm{Cos}\,\Psi(\mathrm{Cos}\,\alpha_2 - 1) - s\,\mathrm{Sin}\,\Psi\,\mathrm{Sin}\,\alpha_2 = p_{21}\,\mathrm{Cos}\,\delta_2$
- $u\,\mathrm{Sin}\,\sigma(\mathrm{Cos}\,\gamma_2 - 1) + u\,\mathrm{Cos}\,\sigma\,\mathrm{Sin}\,\gamma_2 + s\,\mathrm{Sin}\,\Psi(\mathrm{Cos}\,\alpha_2 - 1) + s\,\mathrm{Cos}\,\Psi\,\mathrm{Sin}\,\alpha_2 = p_{21}\,\mathrm{Sin}\,\delta_2$

There are 8 variables in the equations:

3 are given: α_2 , p_{21} and δ_2

Choose 3 variables and solve for 2.

2 Cases to be considered:

Choose σ , γ_2 and Ψ solve for u and s

Choose γ_2 , Ψ and s solve for u and σ

2-Position Procedure – Dyad 2: Case 1

Choose **σ** , γ_2 and **Ψ** then solve for ***u*** and ***s***

Let:

$A = \cos\sigma\,(\cos\gamma_2 - 1) - \sin\sigma \sin\gamma_2$

$B = \cos\Psi(\cos\alpha_2 - 1) - \sin\Psi \sin\alpha_2$

$C = p_{21}\cos\delta_2$

$D = \sin\sigma(\cos\gamma_2 - 1) + \cos\sigma\sin\gamma_2$

$E = \sin\Psi(\cos\alpha_2 - 1) + \cos\Psi\sin\alpha_2$

$F = p21\sin\delta2$

Substitute in the equations below:

$u \cos\sigma(\cos\gamma_2 - 1) - u \sin\sigma \sin\gamma_2 + s \cos\Psi(\cos\alpha_2 - 1) - s \sin\Psi \sin\alpha_2 = p_{21} \cos\delta_2$

$u \sin\sigma(\cos\gamma_2 - 1) + u \cos\sigma \sin\gamma_2 + s \sin\Psi(\cos\alpha_2 - 1) + s \cos\Psi \sin\alpha_2 = p_{21} \sin\delta_2$

$$A\,\mathbf{u} + B\,\mathbf{s} = C$$

$$D\,\mathbf{u} + E\,\mathbf{s} = F$$

Solve:

$$u = (C\,E - B\,F)/(A\,E - B\,D)$$

$$s = (A\,F - C\,D)/(A\,E - B\,D)$$

2-Position Procedure – Dyad 2: Case 2

Choose γ_2 , Ψ and s then solve for u and σ

Let:

$U_{1x} = u\,\text{Cos}\sigma$ $\quad$ $U_{1y} = u\,\text{Sin}\sigma$ $\quad$ $S_{1x} = s\,\text{Cos}\Psi$ $\quad$ $S_{1y} = s\,\text{Sin}\Psi$

$A = \text{Cos}\gamma_2 - 1$ $\quad$ $B = \text{Sin}\gamma_2$ $\quad$ $C = \text{Cos}\alpha_2 - 1$

$D = \text{Sin}\alpha_2$ $\quad$ $E = p_{21}\text{Cos}\delta_2$ $\quad$ $F = p_{21}\text{Sin}\delta_2$

Substitute in the equations below:

$$u\,\text{Cos}\sigma(\text{Cos}\gamma_2 - 1) - u\,\text{Sin}\sigma\,\text{Sin}\gamma_2 + s\,\text{Cos}\Psi(\text{Cos}\alpha_2 - 1) - s\,\text{Sin}\Psi\,\text{Sin}\alpha_2 = p_{21}\,\text{Cos}\delta_2$$
$$u\,\text{Sin}\sigma(\text{Cos}\gamma_2 - 1) + u\,\text{Cos}\sigma\,\text{Sin}\gamma_2 + s\,\text{Sin}\Psi(\text{Cos}\alpha_2 - 1) + s\,\text{Cos}\Psi\,\text{Sin}\alpha_2 = p_{21}\,\text{Sin}\delta_2$$

$$AU_{1x} - B\,U_{1y} + C\,S_{1x} - D\,S_{1y} = E$$

$$AU_{1y} + B\,U_{1x} + C\,S_{1y} + D\,S_{1x} = F$$

Solve:

$$U_{1x} = -\,(A(-\,CS_{1x} + DS_{1y} + E) + B(-\,CS_{1y} - DS_{1x} + F))/2A$$
$$U_{1y} = -\,(A(-\,CS_{1y} - DS_{1x} + F) + B(CS_{1x} - DS_{1y} - E))/2A$$

2-Position Procedure

Solution:

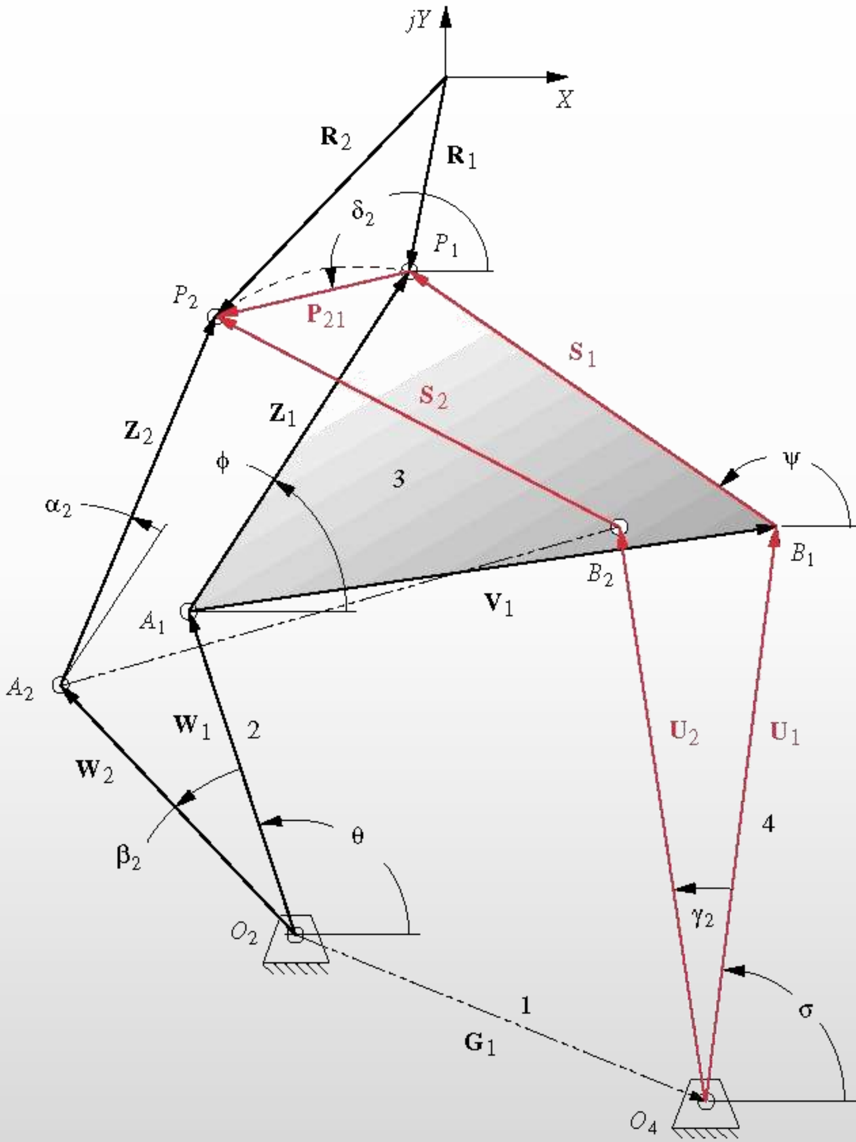
Solve for 1st dyad: WZ

Solve for 2nd dyad: US

Use geometry to solve other details

- Position of the pivots (O_1 and O_2)
- Length of Link1

(analytical: $\mathbf{G_1}=\mathbf{W_1}+\mathbf{V_1}-\mathbf{U_1}$, With $\mathbf{V_1}=\mathbf{Z_1}-\mathbf{S_1}$)



3-Position Procedure

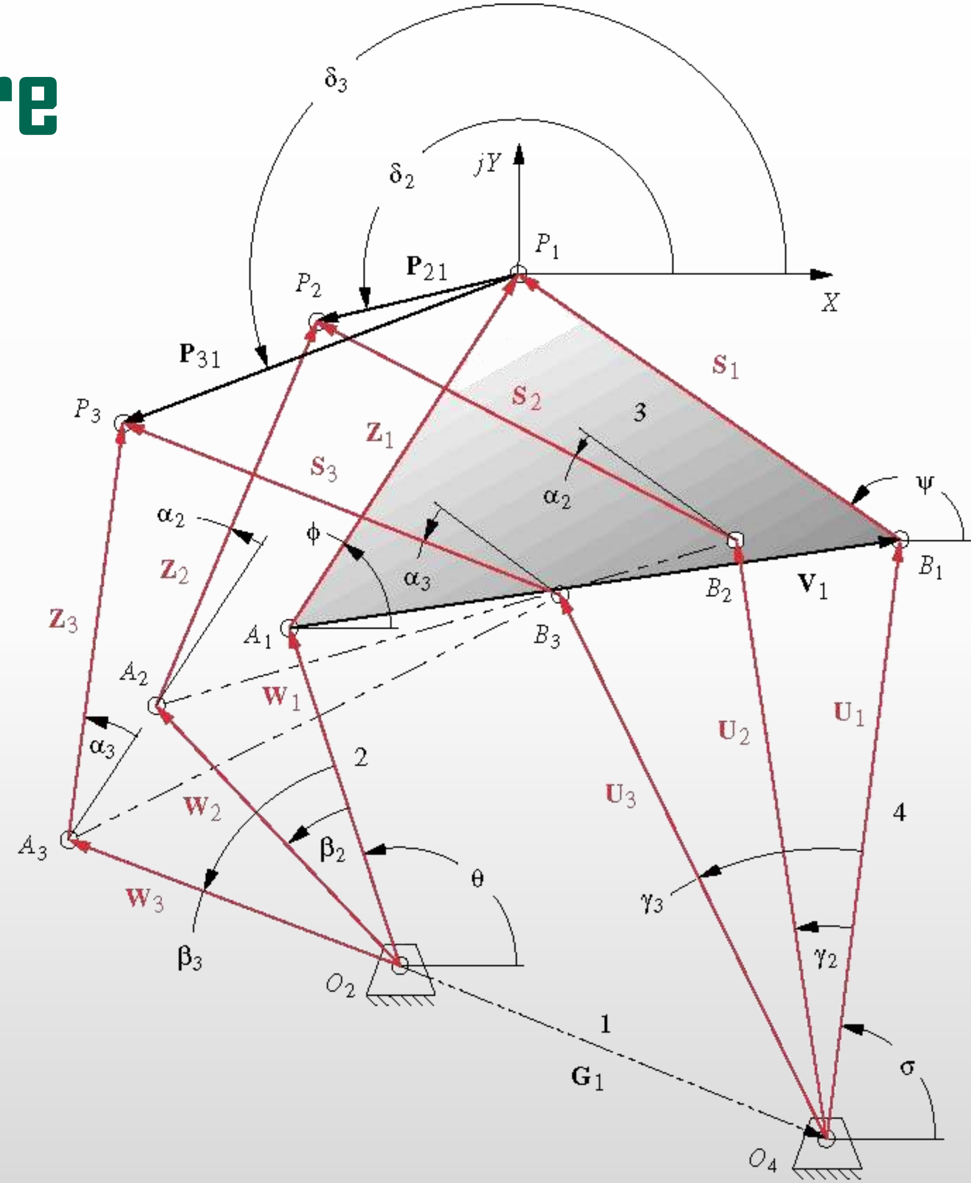
Use the same used for 2-position.

You will have 4 dyads instead of 2

Solution:

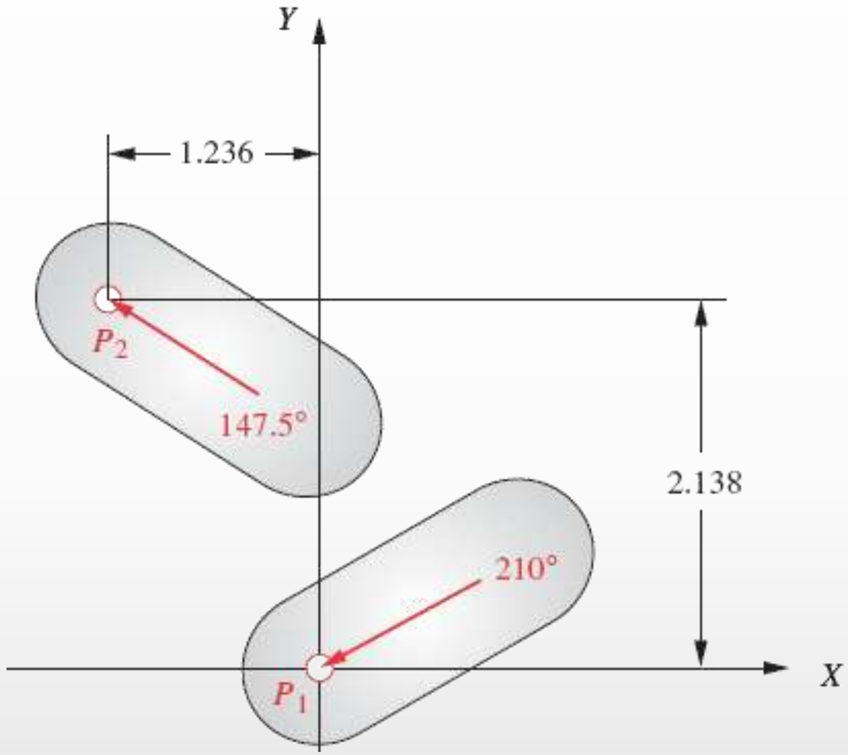
Design of Machinery 5th Edition

Norton p.223–228



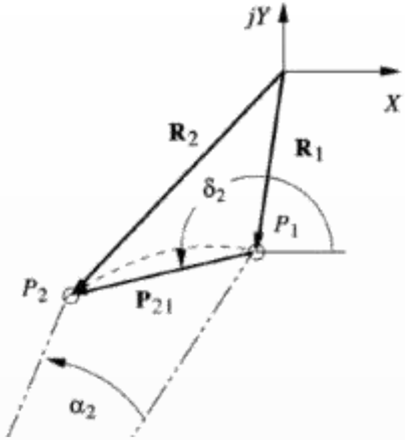
Problem: 5-8 p.255

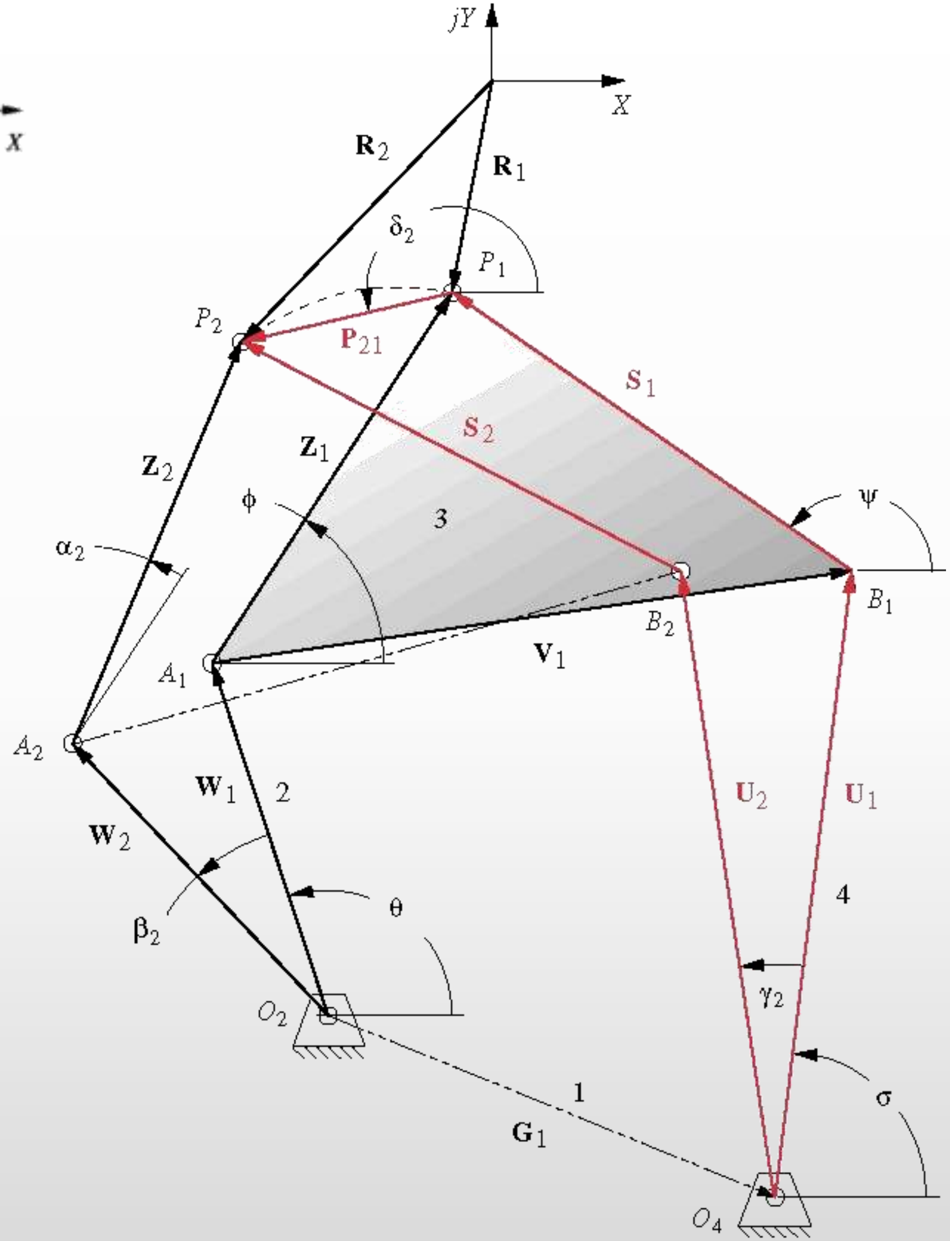
- Design a linkage to carry the body in the figure below through the two positions P1 and P2 at the angles shown in the figure.
- Use analytical synthesis without regard for the fixed pivots.



$$1^{st}\ \text{Dyad} \quad z = 1.075, \phi = 204.4^0, \beta_2 = -27^0$$

$$2^{nd}\ \text{Dyad} \quad s = 1.24, \psi = 74^0, \gamma_2 = -40^0$$


jY
X
R2
R1
δ2
P1
P2
P21
α2


jY
X
R2
R1
δ2
P1
P2
P21
S1
S2
Z2
Z1
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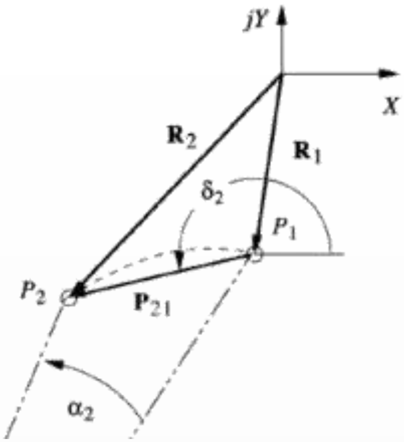
3
ψ
α2
B2
B1
V1
A1
A2
W1
2
W2
U2
U1
θ
4
β2
γ2
O2
σ
1
G1
O4

Solution:



1. Solve the problem using case 2 and the previously generated equations.

2. Define the position vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ and the vector $\mathbf{P}_{21}$ using Figure 5-1 and equation 5.1.

$$R1 := \begin{pmatrix} P_{1x} \\ P_{1y} \end{pmatrix} \qquad R2 := \begin{pmatrix} P_{2x} \\ P_{2y} \end{pmatrix} \qquad \begin{pmatrix} P_{21x} \\ P_{21y} \end{pmatrix} := R2 - R1 \qquad P_{21x} = -1.236$$

$$P_{21y} = 2.138$$

$$p_{21} := \sqrt{P_{21x}{}^2 + P_{21y}{}^2} \qquad p_{21} = 2.470$$

3. From the trigonometric relationships given in Figure 5-1, determine α_2 and δ_2.

$$\alpha_2 := \theta_{P2} - \theta_{P1} \qquad \alpha_2 = -62.500\ deg$$

$$\delta_2 := atan2(P_{21x}, P_{21y}) \qquad \delta_2 = 120.033\ deg$$

Solution:

4. Solve for the **WZ** dyad using equations 5.8.

$Z_{1x} := z \cdot cos(\phi)$ $\quad Z_{1x} = -0.979$ $\quad Z_{1y} := z \cdot sin(\phi)$ $\quad Z_{1y} = -0.444$

$A := cos(\beta_2) - 1$ $\quad A = -0.109$ $\quad D := sin(\alpha_2)$ $\quad D = -0.887$

$B := sin(\beta_2)$ $\quad B = -0.454$ $\quad E := p_{21} \cdot cos(\delta_2)$ $\quad E = -1.236$

$C := cos(\alpha_2) - 1$ $\quad C = -0.538$ $\quad F := p_{21} \cdot sin(\delta_2)$ $\quad F = 2.138$

$$W_{1x} := \frac{A \cdot (-C \cdot Z_{1x} + D \cdot Z_{1y} + E) + B \cdot (-C \cdot Z_{1y} - D \cdot Z_{1x} + F)}{-2 \cdot A} \qquad W_{1x} = -1.462$$

$$W_{1y} := \frac{A \cdot (-C \cdot Z_{1y} - D \cdot Z_{1x} + F) + B \cdot (C \cdot Z_{1x} - D \cdot Z_{1y} - E)}{-2 \cdot A} \qquad W_{1y} = -3.367$$

$$w := \sqrt{W_{1x}^{\ 2} + W_{1y}^{\ 2}} \qquad w = 3.670$$

$$\theta := atan2(W_{1x}, W_{1y}) \qquad \theta = 246.528\ deg$$

Solution:

5. Solve for the **US** dyad using equations 5.12.

$$S_{1x} := s \cdot cos(\psi) \qquad S_{1x} = 0.342 \qquad S_{1y} := s \cdot sin(\psi) \qquad S_{1y} = 1.192$$

$$A := cos(\gamma_2) - 1 \qquad A = -0.234 \qquad D := sin(\alpha_2) \qquad D = -0.887$$

$$B := sin(\gamma_2) \qquad B = -0.643 \qquad E := p_{21} \cdot cos(\delta_2) \qquad E = -1.236$$

$$C := cos(\alpha_2) - 1 \qquad C = -0.538 \qquad F := p_{21} \cdot sin(\delta_2) \qquad F = 2.138$$

$$U_{1x} := \frac{A \cdot (-C \cdot S_{1x} + D \cdot S_{1y} + E) + B \cdot (-C \cdot S_{1y} - D \cdot S_{1x} + F)}{-2 \cdot A} \qquad U_{1x} = -3.180$$

$$U_{1y} := \frac{A \cdot (-C \cdot S_{1y} - D \cdot S_{1x} + F) + B \cdot (C \cdot S_{1x} - D \cdot S_{1y} - E)}{-2 \cdot A} \qquad U_{1y} = -4.439$$

$$u := \sqrt{U_{1x}^{\,2} + U_{1y}^{\,2}} \qquad u = 5.461$$

$$\sigma := atan2(U_{1x}, U_{1y}) \qquad \sigma = 234.381\, deg$$

Solution:

6. Solve for links 3 and 1 using the vector definitions of **V** and **G**.

Link 3:

$$V_{1x} := z\cdot cos(\phi) - s\cdot cos(\psi) \qquad V_{1x} = -1.321$$

$$V_{1y} := z\cdot sin(\phi) - s\cdot sin(\psi) \qquad V_{1y} = -1.636$$

$$\theta_3 := atan2(V_{1x}, V_{1y}) \qquad \theta_3 = 231.086\,deg$$

$$v := \sqrt{V_{1x}^{\ 2} + V_{1y}^{\ 2}} \qquad v = 2.103$$

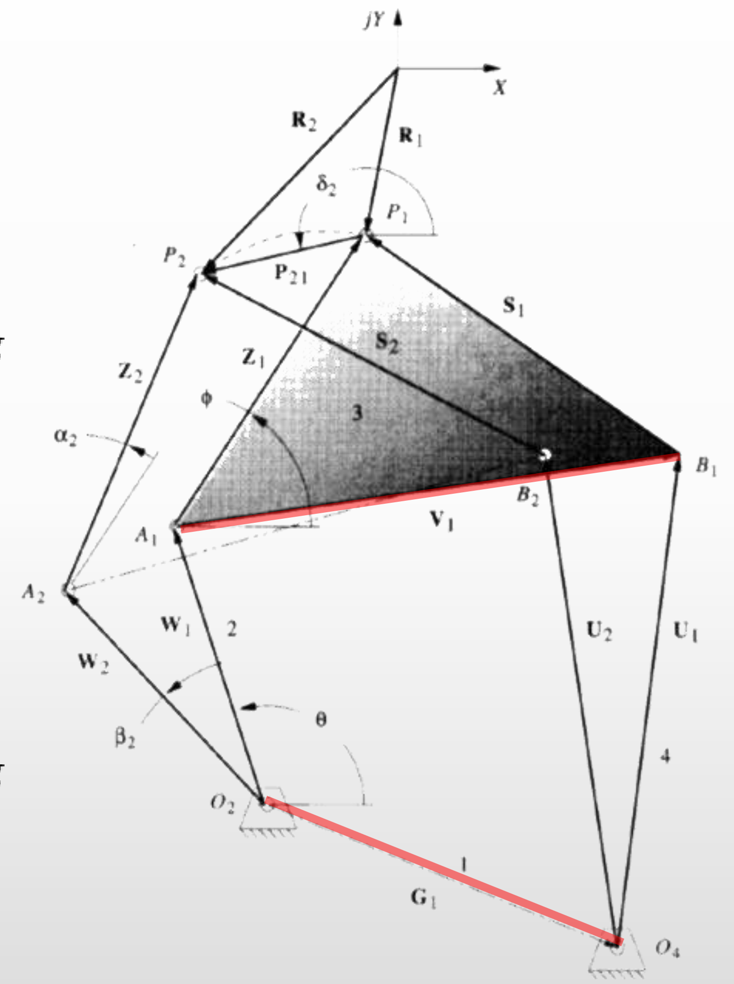
Link 1:

$$G_{1x} := w\cdot cos(\theta) + v\cdot cos(\theta_3) - u\cdot cos(\sigma) \qquad G_{1x} = 0.398$$

$$G_{1y} := w\cdot sin(\theta) + v\cdot sin(\theta_3) - u\cdot sin(\sigma) \qquad G_{1y} = -0.564$$

$$\theta_1 := atan2(G_{1x}, G_{1y}) \qquad \theta_1 = -54.796\,deg$$

$$g := \sqrt{G_{1x}^{\ 2} + G_{1y}^{\ 2}} \qquad g = 0.690$$



Solution:

7. Determine the initial and final values of the input crank with respect to the vector **G**.

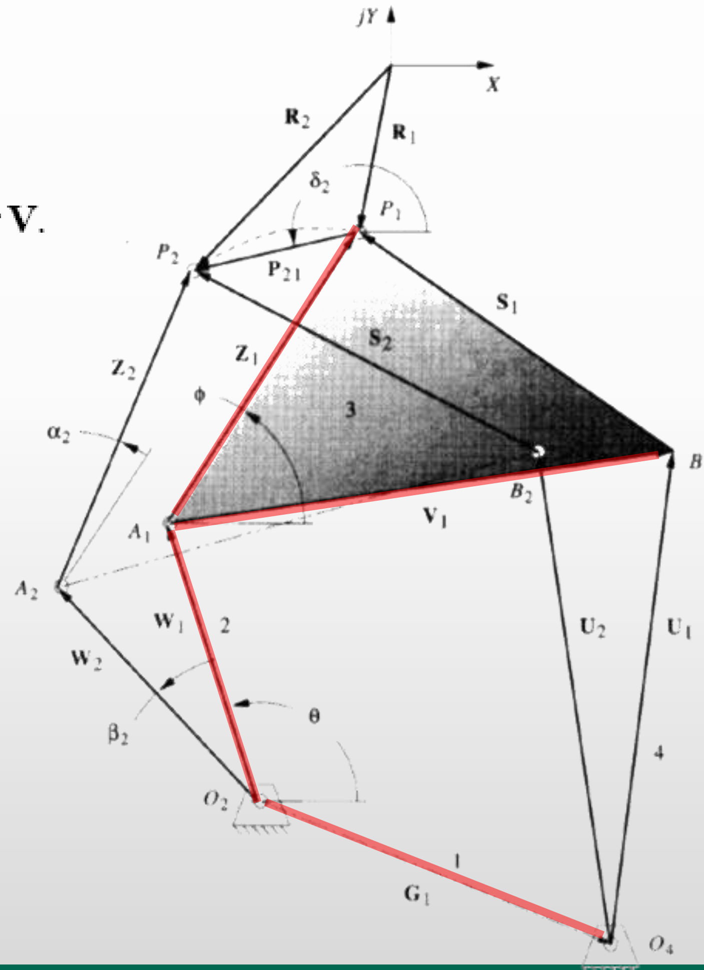
$\theta_{2i} := \theta - \theta_1 \qquad \theta_{2i} = 301.323\,deg$

$\theta_{2f} := \theta_{2i} + \beta_2 \qquad \theta_{2f} = 274.323\,deg$

8. Define the coupler point with respect to point A and the vector **V**.

$r_p := z \qquad \delta_p := \phi - \theta_3$

$r_p = 1.075 \qquad \delta_p = -26.686\,deg$



Solution:

9. Locate the fixed pivots in the global frame using the vector definitions

$\rho_1 := atan2(P_{1x}, P_{1y})$ $\rho_1 = 180.000\, deg$

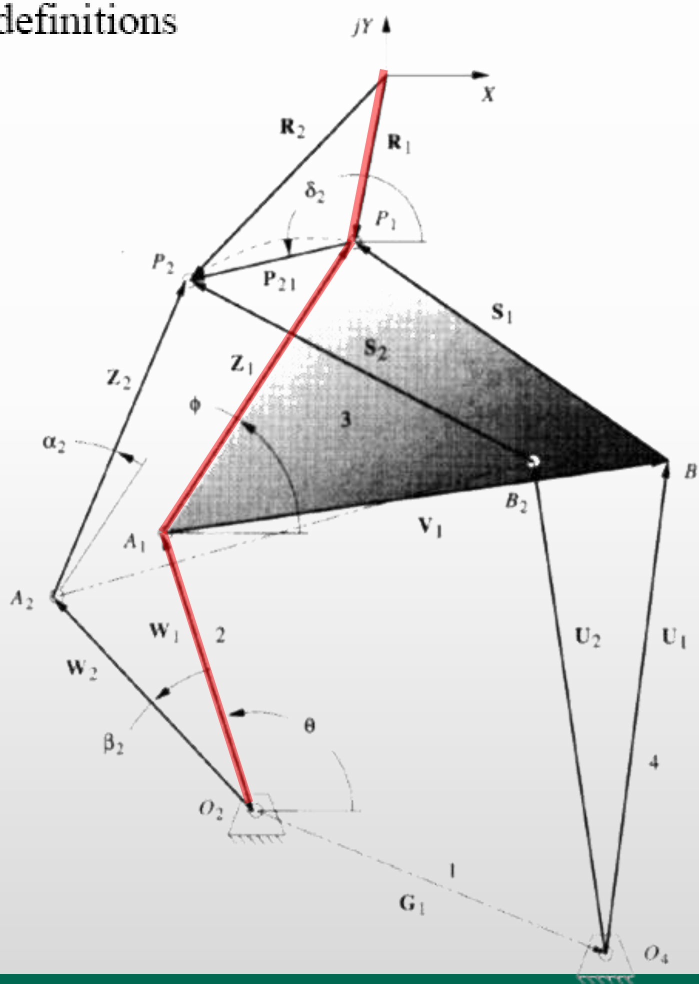
$R_1 := \sqrt{P_{1x}^2 + P_{1y}^2}$ $R_1 = 0.000$

$O_{2x} := R_1 \cdot cos(\rho_1) - z \cdot cos(\phi) - w \cdot cos(\theta)$ $O_{2x} = 2.441$

$O_{2y} := R_1 \cdot sin(\rho_1) - z \cdot sin(\phi) - w \cdot sin(\theta)$ $O_{2y} = 3.811$

$O_{4x} := R_1 \cdot cos(\rho_1) - s \cdot cos(\psi) - u \cdot cos(\sigma)$ $O_{4x} = 2.838$

$O_{4y} := R_1 \cdot sin(\rho_1) - s \cdot sin(\psi) - u \cdot sin(\sigma)$ $O_{4y} = 3.247$



Solution:

10. Determine the rotation angle of the fourbar frame with respect to the global frame (angle from the global X axis to the line O_2O_4.

$$\theta_{rot} := atan2\left[\left(O_{4x} - O_{2x}\right),\left(O_{4y} - O_{2y}\right)\right] \qquad \theta_{rot} = -54.796\,deg$$

11. Determine the Grashof condition.

$$Condition(a,b,c,d) := \left\| \begin{array}{l} S \leftarrow min(a,b,c,d) \\ L \leftarrow max(a,b,c,d) \\ SL \leftarrow S + L \\ PQ \leftarrow a + b + c + d - SL \\ return\ \text{"Grashof"}\ \ if\ \ SL < PQ \\ return\ \text{"Special Grashof"}\ \ if\ \ SL = PQ \\ return\ \text{"non-Grashof"}\ \ otherwise \end{array} \right.$$

Solution:

12. **DESIGN SUMMARY**

Link 2:	$w = 3.670$	$\theta = 246.528\ deg$
Link 3:	$v = 2.103$	$\theta_3 = 231.086\ deg$
Link 4:	$u = 5.461$	$\sigma = 234.381\ deg$
Link 1:	$g = 0.690$	$\theta_1 = -54.796\ deg$
Coupler:	$r_p = 1.075$	$\delta_p = -26.686\ deg$
Crank angles:		
	$\theta_{2i} = 301.323\ deg$	
	$\theta_{2f} = 274.323\ deg$	

Solution:

13. Draw the linkage, using the link lengths, fixed pivot positions, and angles above, to verify the design.